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van Velzen, S.; Hamers, H.J.M.; Norde, H.W.

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By Bas van Velzen, Herbert Hamers, Henk Norde

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A note on permutationally convex games

Bas van Velzen[a] Herbert Hamers[a, b] Henk Norde[a]

Abstract

In this paper we generalise marginal vectors and permutational convexity. We show that if a game is generalised permutationally convex, then the corresponding generalised marginal vector is a core element. Furthermore we refine the concept of permutational convexity and show that this refinement yields a sufficient condition for the corresponding marginal vector to be a core element. Finally, we prove that permutational convexity is equivalent to a restricted set of inequalities and that if a game is permutationally convex with respect to an order, then it is permutationally convex with respect to a related order as well.

JEL Classification: C71
Keywords: Cooperative game theory, marginal vectors, permutational convexity.

1 Introduction

Permutational convexity (concavity) was first introduced in [2]. In that paper it is shown that permutational convexity yields a sufficient condition for a corresponding marginal vector to be a core element. By applying this result to minimum cost spanning tree games, it is shown that specific marginal vectors are core elements.

The approach of [2] is adopted by, e.g., [1], [3], [4] and [5]. In [1] the permutational convexity of minimum cost spanning forest games is shown, and in [3] the non-emptiness of the core of holding cost games is shown with the use of permutational concavity. The existence of core elements for two types of sequencing games is shown in [4] and [5] with the aid of permutational convexity.

In this paper we generalise the concept of marginal vector to trees. By generalising permutational convexity we obtain a sufficient condition for the corresponding generalised marginal vector to be a core element. Furthermore we refine permutational convexity and show that this refinement is still sufficient for core-membership of the corresponding marginal vector. Besides that we show that permutational convexity is equivalent to a restricted set of inequalities. In particular we reduce the number of permutational convexity inequalities by a factor two. Finally we show that if a game is permutationally convex with respect to an order, then it is permutationally convex with respect to the last neighbour of this order as well.

The remainder of this paper is organised as follows. In Section 2 we recall some basic concepts of cooperative game theory. In Section 3 we prove that generalised permutational convexity is sufficient for a generalised marginal vector to be a core element. Section 4 is dedicated to a refinement of permutational convexity. Finally, in Section 5 we show that permutational convexity is equivalent to a restricted set of inequalities and that if a game is permutationally convex with respect to an order, then it is permutationally convex with respect to the last neighbour of this order as well.

[a]CentER and Department of Econometrics and OR, Tilburg University, P.O.Box 90153, 5000 LE Tilburg, The Netherlands.

[b]Corresponding author. Email: h.j.m.hamers@uvt.nl

2 Preliminaries

In this section we recall some concepts from cooperative game theory.

A *TU-game* (N, v) consists of a finite set of players $N = \{1, \ldots, |N|\}$ and a map $v : 2^N \to \mathbb{R}$ that expresses the worth of each coalition. By convention we assume that $v(\emptyset) = 0$. The class of TU-games with player set N will be denoted by TU^N. The *core* of $v \in TU^N$ is the set of efficient payoff vectors for which no coalition has an incentive to split off from the grand coalition, i.e. $C(v) = \{x \in \mathbb{R}^N : \sum_{i\in N} x_i = v(N), \sum_{i\in S} x_i \geq v(S) \text{ for all } S \subseteq N\}$. The core of a game can be empty. An *order* on N is a bijection $\sigma : \{1, \ldots, |N|\} \to N$. The set of all orders is given by $\Pi(N)$. Let $\sigma \in \Pi(N)$. The *i-th neighbour* σ_i of σ is obtained from σ by interchanging the players at the i-th and $(i+1)$-st position of σ. So $\sigma_i(j) = \sigma(j)$ for all $j \in \{1, \ldots, |N|\}$, $j \neq i, i+1$, $\sigma_i(i) = \sigma(i+1)$ and $\sigma_i(i+1) = \sigma(i)$.

Let $\sigma \in \Pi(N)$. Define $[\sigma(i), \sigma] = \{\sigma(j) : j \leq i\}$ to be the set of all predecessors of $\sigma(i)$, for all $i \in \{1, \ldots, |N|\}$. Additionally we define, for notational simplicity, $[\sigma(0), \sigma] = \emptyset$. Let $v \in TU^N$. The *marginal vector* $m^\sigma(v)$ is given by $m^\sigma_{\sigma(i)}(v) = v([\sigma(i), \sigma]) - v([\sigma(i-1), \sigma])$ for each $1 \leq i \leq |N|$.

The game $v \in TU^N$ is said to be *permutationally convex with respect to* $\sigma \in \Pi(N)$ if

$$v([\sigma(i), \sigma] \cup S) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup S) + v([\sigma(i), \sigma]), \tag{1}$$

for all $0 \leq i < k \leq |N| - 1$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. If (N, v) is permutationally convex with respect to σ, then σ is also called permutationally convex for (N, v). The following theorem, proved in [2], shows that if a TU-game is permutationally convex with respect to $\sigma \in \Pi(N)$, then the corresponding marginal vector is a core element. We remark that the reverse of this theorem is not true in general.

Theorem 1 ([2]). Let $v \in TU^N$. If $\sigma \in \Pi(N)$ is a permutationally convex order for (N, v), then $m^\sigma(v) \in C(v)$.

Let $v \in TU^N$. Then checking if an order $\sigma \in \Pi(N)$ is permutationally convex, requires the checking of many inequalities. In fact, for each $0 \leq i < k \leq |N| - 1$, there are precisely $2^{|N|-k} - 1$ choices of $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. Hence, for each $0 \leq i < k \leq |N| - 1$, there are precisely $2^{|N|-k} - 1$ permutational convexity inequalities. So in total there are

$$\sum_{i=0}^{|N|-2} \sum_{k=i+1}^{|N|-1} [2^{|N|-k} - 1] = \sum_{i=0}^{|N|-2} [2^{|N|-i} - 2 - (|N| - i - 1)]$$

$$= 2^{|N|+1} - 4 - 2(|N| - 1) - \frac{1}{2}(|N| - 1)|N| = 2^{|N|+1} - 2 - \frac{1}{2}|N|^2 - 1\frac{1}{2}|N|$$

inequalities.

3 Generalised marginal vectors and generalised permutational convexity

In this section we introduce a generalisation of marginal vectors. By introducing a generalisation of permutational convexity we find sufficient conditions for the corresponding generalised marginal vector to be a core element.

Let $G = (N, E)$ be a directed rooted tree with the arcs pointed towards the root. Note that each node (player) has precisely one follower, except for the root (player coinciding with the root). The set of predecessors of $j \in N$ with respect to G is the set $P(j) = \{i \in N :$ there exists a directed path in G from i to $j\}$. Note that $j \in P(j)$. For notational simplicity we define $P(0) = \emptyset$. Let $D(j) = \{i \in N : (i,j) \in E\}$ be the set of direct predecessors of j. Let $v \in TU^N$ and define the payoff vector $h^G(v)$ by $h^G_j(v) = v(P(j)) - \sum_{i \in D(j)} v(P(i))$ for all $j \in N$. Note that if G is a directed chain, then $h^G(v)$ corresponds to a marginal vector. Therefore we call $h^G(v)$ the *generalised marginal vector with respect to* G. The following example illustrates the concept of generalised marginal vectors.

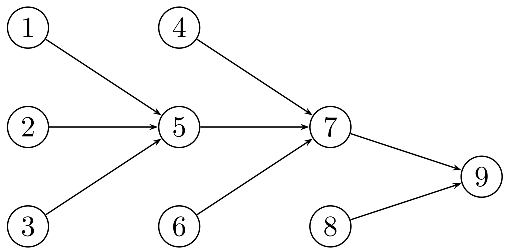

Figure 1: The directed rooted tree G.

Example 1. Consider the directed rooted tree $G = (N, E)$ of Figure 1 and let $v \in TU^N$. Then, for instance, $h^G_1(v) = v(\{1\})$, $h^G_7(v) = v(\{1,2,3,4,5,6,7\}) - v(\{1,2,3,5\}) - v(\{4\}) - v(\{6\})$ and $h^G_9(v) = v(N) - v(\{1,2,3,4,5,6,7\}) - v(\{8\})$. $\diamond$

A game $v \in TU^N$ is called *generalised permutationally convex with respect to* $G = (N, E)$ if for all $M \subseteq N$, with $P(i) \cap P(j) = \emptyset$ for all $i, j \in M$, $i \neq j$,

$$v(\bigcup_{i \in M} P(i)) \leq \sum_{i \in M} v(P(i)), \tag{2}$$

and for all $j, k \in N$ with $(k, j) \in E$, $I \subseteq P(k)$ with $P(l) \subseteq I$ for all $l \in I$, and $S \subseteq N \backslash P(k)$ with $j \in S$,

$$v(I \cup S) + v(P(k)) \leq v(P(k) \cup S) + v(I). \tag{3}$$

We remark that the term generalised permutational convex is slightly misleading, because if $G = (N, E)$ is a directed chain, then the concepts of permutational convexity and generalised permutational convexity do not coincide. In fact, if $G = (N, E)$ is a directed chain, then all inequalities of (2) disappear, but the inequalities of (3) boil down to a (proper) subset of the set of permutational convexity inequalities. Hence, generalised permutational convexity is weaker for directed chains than permutational convexity. In the following example we illustrate generalised permutational convexity inequalities.

Example 2. Consider the tree from Figure 1. If $v \in TU^N$ is generalised permutationally convex with respect to G, then it follows from (2), taking $M = \{5, 8\}$, that $v(\{1,2,3,5,8\}) \leq v(\{1,2,3,5\}) + v(\{8\})$. Similarly, it follows from (3), with $k = 7$, $j = 9$, $I = \{1,2,3,5\}$ and $S = \{8, 9\}$, that $v(\{1,2,3,5,8,9\}) + v(\{1,2,3,4,5,6,7\}) \leq v(N) + v(\{1,2,3,5\})$. $\diamond$

The following theorem is a generalisation of Theorem 1. It states that if a game is generalised permutationally convex with respect to a directed rooted tree, then the corresponding generalised marginal vector is a core element. In the proof of the theorem we make use of the following notations and definitions.

For any $W \subseteq N$, let $B(W) = \{i \in W : P(i) \cap W = \{i\}\}$ be the set of players in W without any predecessors in W, except themselves. Furthermore define $A(W) = \bigcup_{j \in B(W)} D(j)$. Let $Q \subseteq N$.[a]

[a]The following sets we introduce all depend on Q. For the sake of notational convenience we omit this dependency in notation.

Then

$$E(Q) = \{i \in Q : i \notin P(j) \text{ for all } j \in Q \backslash \{i\}\}$$

is the set of players in Q without any followers in Q. Furthermore, for each $j \in Q$,

$$DP(j) = \{i \in Q \cap P(j) : i \neq j, i \in P(j) \text{ and there is no } k \in Q \cap P(j), k \neq j, \text{ with } i \in P(k)\}$$

is the set of players in Q directly preceding j. Finally, let $l \in Q$ and consider $D(l)$. Then $i \in D(l)$ can be such that $i \in Q$, such that $i \notin Q$ and $P(i) \cap Q = \emptyset$, or such that $i \notin Q$ but $P(i) \cap Q \neq \emptyset$. Therefore we partition $D(l)$ into three subsets $D_1(l)$, $D_2(l)$ and $D_3(l)$ in the following way:

$$\begin{aligned} D_1(l) &= \{i \in D(l) : i \in Q\}, \\ D_2(l) &= \{i \in D(l) : P(i) \cap Q = \emptyset\}, \\ D_3(l) &= \{i \in D(l) : i \notin Q \text{ but } P(i) \cap Q \neq \emptyset\}. \end{aligned}$$

Now note that $D_1(l) = D(l) \cap DP(l)$, and that $D_3(l) = \{i \in D(l) : i \notin DP(l) \text{ but } DP(l) \cap P(i) \neq \emptyset\}$. By definition of $D_3(l)$, for each $i \in D_3(l)$ there is at least one $j \in P(i)$ with $j \in DP(l)$. Now define $M_l(i)$ as the set containing all these players, i.e.

$$M_l(i) = DP(l) \cap P(i).$$

Finally, let for each $i \in D_3(l)$, $R(i)$ be the set of players that are followed by a player in $M_l(j)$, i.e.

$$R(i) = \bigcup_{j \in M_l(i)} P(j).$$

We illustrate these sets in the following example.

Example 3. Let $G = (N, E)$ be the directed tree depicted in Figure 1. Then $B(\{5, 7, 8\}) = \{5, 8\}$ and $A(\{5, 7, 8\}) = \{1, 2, 3\}$. Let $Q = \{1, 2, 4, 7, 8\}$. Then $E(Q) = \{7, 8\}$, $DP(7) = \{1, 2, 4\}$. Furthermore, $D(7) = \{4, 5, 6\}$, $D_1(7) = D(7) \cap \{1, 2, 4\} = \{4\}$, $D_2(7) = \{6\}$ and $D_3(7) = \{5\}$. Finally, $M_7(5) = DP(7) \cap P(5) = \{1, 2\}$ and $R(5) = \{1, 2\}$.

The following theorem shows that generalised permutational convexity is a sufficient condition for the generalised marginal vector to be a core element. In the proof of this theorem we make use of several technical lemmas that are stated and proved in the Appendix.

Theorem 2. Let $v \in TU^N$ and let $G = (N, E)$ be a directed rooted tree with the arcs pointed towards the root. If (N, v) is generalised permutationally convex with respect to G, then $h^G(v) \in C(v)$.

Proof: Since $h^G(v)$ is efficient by definition, we only need to show for each $Q \subseteq N$ that $\sum_{i \in Q} h_i^G(v) \geq v(Q)$. Let $Q \subseteq N$. We show $\sum_{i \in Q} h_i^G(v) \geq v(Q)$ in a recursive manner. In fact, we obtain a sequence of coalitions $T_1 \subsetneq T_2 \subsetneq \ldots \subsetneq T_m = Q$ such that

$$\sum_{q \in T_t} h_q^G(v) \geq v(T_t \cup \bigcup_{i \in A(T_t)} P(i)) - \sum_{i \in A(T_t)} v(P(i)) \tag{4}$$

for each $1 \leq t \leq m$. For $t = m$, expression (4) boils down to

$$\sum_{q \in Q} h_q^G(v) \geq v(Q \cup \bigcup_{i \in A(Q)} P(i)) - \sum_{i \in A(Q)} v(P(i)). \tag{5}$$

Now let $A(Q) = \{a_1, \ldots, a_q\}$. Note that for each $a_i \in A(Q)$ there is a $j_i \in Q$ with $(a_i, j_i) \in E$. Hence, it follows from (3), by taking $k = a_i$, $j = j_i$, $I = \emptyset$ and $S = Q \cup \bigcup_{l=1}^{i-1} P(a_l)$, that

$$v(Q \cup \bigcup_{l=1}^{i-1} P(a_l)) + v(P(a_i)) \leq v(Q \cup \bigcup_{l=1}^{i} P(a_l)).$$

By summing over all $1 \leq i \leq q$, and using the telescopic nature of the obtained expression we conclude that

$$v(Q) + \sum_{l=1}^{q} v(P(a_l)) \leq v(Q \cup \bigcup_{l=1}^{i} P(a_l)),$$

or equivalently,

$$v(Q) + \sum_{i \in A(Q)} v(P(i)) \leq v(Q \cup \bigcup_{i \in A(Q)} P(i)).$$

This last expression together with (5) shows that $\sum_{i \in Q} h_i^G(v) \geq v(Q)$.

Now we will first concentrate on our sequence $T_1 \subsetneq T_2 \subsetneq \ldots \subsetneq T_m = Q$. Our sequence starts with $T_1 = E(Q)$. Note that $E(Q) \subseteq Q$. If $E(Q) \neq Q$, then $Q \backslash E(Q) \neq \emptyset$. In particular there is a $k \in Q \backslash E(Q)$ with $k \in DP(l)$ for some $l \in B(E(Q))$. Note that it is even satisfied that $DP(l) \cap E(Q) = \emptyset$. Now set $T_2 = T_1 \cup DP(l)$ and observe that $T_2 \subseteq Q$.

Now we will proceed this sequence in a similar way. To be more precise, each sequence element T_p is constructed from T_{p-1} by taking the union of T_{p-1} with the set of direct predecessors of an element in $B(T_{p-1})$. So if $T_{p-1} \subsetneq Q$, then there is a $k \in Q \backslash T_{p-1}$ with $k \in DP(l)$ for some $l \in B(T_{p-1})$. Observe, by construction of our sequence, that $T_{p-1} \cap DP(l) = \emptyset$. Now set $T_p = T_{p-1} \cup DP(l)$, and observe that both $T_{p-1} \subsetneq T_p$ and $T_p \subseteq Q$ are trivially satisfied. Finally note that if $l \in N$ is such that $T_p = T_{p-1} \cup DP(l)$, then $B(T_p) = (B(T_{p-1}) \backslash \{l\}) \cup DP(l)$.

It remains to show that (4) is indeed satisfied for each $1 \leq t \leq m$. We will show this inequality by induction on t. First, let $t = 1$. Then,

$$\begin{aligned}
\sum_{q \in E(Q)} h_q^G(v) &= \sum_{q \in E(Q)} [v(P(q)) - \sum_{i \in D(q)} v(P(i))] \\
&= \sum_{q \in E(Q)} v(P(q)) - \sum_{q \in E(Q)} \sum_{i \in D(q)} v(P(i)) \\
&\geq v(\bigcup_{q \in E(Q)} P(q)) - \sum_{q \in E(Q)} \sum_{i \in D(q)} v(P(i)) \\
&= v(E(Q) \cup \bigcup_{i \in A(E(Q))} P(i)) - \sum_{i \in A(E(Q))} v(P(i)).
\end{aligned}$$

The inequality follows from (2) with $M = E(Q)$, where we note that $P(i) \cap P(j) = \emptyset$ for all $i, j \in E(Q)$, $i \neq j$. The last equality follows from the third assertion of Lemma 1 and the definition of $A(E(Q))$.

For the induction hypothesis, assume that (4) is satisfied for all $1 \leq t \leq \bar{t}$, with $1 \leq \bar{t} \leq m$. If $\bar{t} = m$, then we are done, so assume that $\bar{t} < m$.

For the induction step, let $t^* = \bar{t}+1$. Let $l \in B(T_{t^*})$ be such that $T_{t^*+1} = T_{t^*} \cup DP(l)$. Then,

$$\begin{aligned}
\sum_{q\in T_{t^*}} h_q^G(v) &= \sum_{q\in T_{t^*-1}} h_q^G(v) + \sum_{q\in DP(l)} h_q^G(v)\\
&\geq v(T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})} P(i)) - \sum_{i\in A(T_{t^*-1})} v(P(i)) + \sum_{q\in DP(l)} [v(P(q)) - \sum_{i\in D(q)} v(P(i))]\\
&= v(T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})} P(i)) - \sum_{i\in A(T_{t^*-1})} v(P(i)) + \sum_{q\in DP(l)} v(P(q))\\
&\quad - \sum_{q\in DP(l)} \sum_{i\in D(q)} v(P(i))\\
&= v(T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})} P(i)) + \sum_{q\in DP(l)} v(P(q)) - \sum_{i\in D(l)} v(P(i)) \qquad (6)\\
&\quad - \sum_{i\in A(T_{t^*})} v(P(i)).
\end{aligned}$$

The inequality is because of our induction hypothesis that (4) is satisfied for all $t \leq \bar{t}$ and the definition of $h^G(v)$. In particular, (4) is satisfied for $t = t^*-1$. The last equality follows from Lemma 3. So showing that (4) holds for $t = t^*$ boils down to showing that (6) exceeds $v(T_{t^*} \cup \bigcup_{i\in A(T_{t^*})} P(i))$. First we slightly rewrite (6). Observe that

$$\begin{aligned}
&v(T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})} P(i)) + \sum_{q\in DP(l)} v(P(q)) - \sum_{i\in D(l)} v(P(i))\\
=&v(T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})} P(i)) + \sum_{q\in DP(l)\backslash D_1(l)} v(P(q)) - \sum_{i\in D_2(l)} v(P(i)) - \sum_{i\in D_3(l)} v(P(i))\\
=&v(\bar{T} \cup \bigcup_{i\in D_2(l)} P(i) \cup \bigcup_{i\in D_3(l)} P(i)) + \sum_{q\in DP(l)\backslash D_1(l)} v(P(q))\\
&- \sum_{i\in D_2(l)} v(P(i)) - \sum_{i\in D_3(l)} v(P(i)),
\end{aligned} \qquad (7)$$

where $\bar{T} = T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})\backslash D(l)} P(i) \cup \bigcup_{i\in D_1(l)} P(i)$. The first equality follows by definition of $D_1(l)$, $D_2(l)$ and $D_3(l)$. The second equality follows from the second assertion of Lemma 4.

We will now find a bound for (7) by repeatedly applying (3). First write $D_2(l) = \{d_1, \ldots, d_r\}$. For each $1 \leq i \leq r$, we want to apply (3) with $k = d_i$, $j = l$, $I = \emptyset$ and $S = \bar{T} \cup \bigcup_{q=1}^{i-1} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)$. So we need to check that $k = d_i$, $j = l$, $I = \emptyset$ and $S = \bar{T} \cup \bigcup_{q=1}^{i-1} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)$ satisfy the conditions accompanying (3). First observe that indeed $(l, d_i) \in E$. Furthermore note that $l \in T_{t^*-1}$ implies $l \in \bar{T} \cup \bigcup_{q=1}^{i-1} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)$. So it remains to check that $\Big(\bar{T} \cup \bigcup_{q=1}^{i-1} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)\Big) \subseteq N\backslash P(d_i)$, or equivalently that $\Big(\bar{T} \cup \bigcup_{q=1}^{i-1} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)\Big) \cap P(d_i) = \emptyset$. First observe that $P(d_q) \cap P(d_i) = \emptyset$ for each $1 \leq q \leq i-1$ and that $P(q) \cap P(d_i) = \emptyset$ for each $q \in D_3(l)$. It remains to check that $\bar{T} \cap P(d_i) = \emptyset$. Now note that $P(q) \cap P(d_i) = \emptyset$ for each $q \in D_1(l)$, and that $T_{t^*} \cap P(d_i) = \emptyset$. Finally, note that $P(q) \cap P(l) = \emptyset$ for each $q \in \bigcup_{p\in A(T_{t^*-1})\backslash D(l)} P(p)$. This implies, using $d_i \in P(l)$, that $P(q) \cap P(d_i) = \emptyset$ for each $q \in \bigcup_{p\in A(T_{t^*-1})\backslash D(l)} P(p)$. So $\bar{T} \cap P(d_i) = \emptyset$ and we conclude that (3) can be applied with $k = d_i$, $j = l$, $I = \emptyset$ and $S = \bar{T} \cup \bigcup_{q=1}^{i-1} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)$. This yields

$$v(\bar{T} \cup \bigcup_{q=1}^{i-1} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)) + v(P(d_i)) \leq v(\bar{T} \cup \bigcup_{q=1}^{i} P(d_q) \cup \bigcup_{q\in D_3(l)} P(q)).$$

By summing over all $1 \leq i \leq r$, and using the telescopic nature of the inequality we obtain

$$v(\bar{T} \cup \bigcup_{q \in D_3(l)} P(q)) + \sum_{q=1}^{r} v(P(d_q)) \leq v(\bar{T} \cup \bigcup_{q=1}^{r} P(d_q) \cup \bigcup_{q \in D_3(l)} P(q)),$$

or equivalently,

$$v(\bar{T} \cup \bigcup_{q \in D_3(l)} P(q)) + \sum_{q \in D_2(l)} v(P(q)) \leq v(\bar{T} \cup \bigcup_{q \in D_2(l)} P(q) \cup \bigcup_{q \in D_3(l)} P(q)).$$

Substituting this last inequality in (7) yields

$$\begin{aligned} & v(\bar{T} \cup \bigcup_{i \in D_2(l)} P(i) \cup \bigcup_{i \in D_3(l)} P(i)) + \sum_{q \in DP(l) \setminus D_1(l)} v(P(q)) \\ & - \sum_{i \in D_2(l)} v(P(i)) - \sum_{i \in D_3(l)} v(P(i)) \\ \geq \; & v(\bar{T} \cup \bigcup_{i \in D_3(l)} P(i)) + \sum_{q \in DP(l) \setminus D_1(l)} v(P(q)) - \sum_{i \in D_3(l)} v(P(i)). \end{aligned} \tag{8}$$

Now write $D_3(l) = \{e_1, \ldots, e_s\}$ and let $1 \leq i \leq s$. Then observe that $(e_i, l) \in E$, $R(e_i) \subseteq P(e_i)$ and $P(l) \in R(e_i)$ for each $l \in R(e_i)$. So we can apply (3) with $k = e_i$, $j = l$, $I = R(e_i)$ and $S = \bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)$ if it is satisfied that $l \in \bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)$ and that $\left(\bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)\right) \subseteq N \setminus P(e_i)$ and $l \in T_{t^*-1}$. First note that $l \in T_{t^*-1}$ implies $l \in \bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)$. So it remains to show that $\left(\bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)\right) \subseteq N \setminus P(e_i)$, or equivalently that $\left(\bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)\right) \cap P(e_i) = \emptyset$. Now note that $P(e_q) \cap P(e_i) = \emptyset$ for each $1 \leq q \leq i-1$, and $R(e_q) \cap P(e_i)$ for each $i+1 \leq q \leq s$. It remains to check that $\bar{T} \cap P(e_i) = \emptyset$. Note that $P(q) \cap P(e_i)$ for each $q \in D_1(l)$ and that $T_{t^*} \cap P(e_i) = \emptyset$. Finally, note that $P(q) \cap P(l) = \emptyset$ for each $q \in \bigcup_{p \in A(T_{t^*-1}) \setminus D(l)} P(p)$. This implies, using $e_i \in P(l)$, that $P(q) \cap P(e_i) = \emptyset$ for each $q \in \bigcup_{p \in A(T_{t^*-1}) \setminus D(l)} P(p)$. So $\bar{T} \cap P(e_i) = \emptyset$ and we conclude that (3) can be applied with $k = e_i$, $j = l$, $I = R(e_i)$ and $S = \bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)$. This yields

$$v(\bar{T} \cup \bigcup_{q=1}^{i-1} P(e_q) \cup \bigcup_{q=i}^{s} R(e_q)) + v(P(e_i)) \leq v(\bar{T} \cup \bigcup_{q=1}^{i} P(e_q) \cup \bigcup_{q=i+1}^{s} R(e_q)) + v(R(e_i)).$$

By summing over all $1 \leq i \leq s$ and using the telescopic nature of the inequality we find that

$$v(\bar{T} \cup \bigcup_{i=1}^{s} R(e_i)) + \sum_{i=1}^{s} v(P(e_i)) \leq v(\bar{T} \cup \bigcup_{i=1}^{s} P(e_i)) + \sum_{i=1}^{s} v(R(e_i)),$$

or equivalently,

$$v(\bar{T} \cup \bigcup_{q \in D_3(l)} R(q)) + \sum_{q \in D_3(l)} v(P(q)) \leq v(\bar{T} \cup \bigcup_{q \in D_3(l)} P(q)) + \sum_{q \in D_3(l)} v(R(q)). \tag{9}$$

By substituting this inequality in (8) we obtain

$$\begin{aligned}
& v(\bar{T} \cup \bigcup_{i \in D_3(l)} P(i)) + \sum_{q \in DP(l)\backslash D_1(l)} v(P(q)) - \sum_{i \in D_3(l)} v(P(i)) \\
\geq\ & v(\bar{T} \cup \bigcup_{q \in D_3(l)} R(q)) + \sum_{q \in DP(l)\backslash D_1(l)} v(P(q)) - \sum_{q \in D_3(l)} v(R(q)) \\
\geq\ & v(\bar{T} \cup \bigcup_{q \in D_3(l)} R(q)), \\
=\ & v(T_{t^*} \cup \bigcup_{i \in A(T_{t^*})} P(i)).
\end{aligned}$$

The first inequality is due to (9) and the second because of Lemma 6. The equality is due to Lemma 7. □

Let $G = (N, E)$ be a directed chain and let $v \in TU^N$. We noted before that in this case $h^G(v)$ corresponds to a marginal vector, and that the inequalities of (3) boil down to permutational convexity inequalities. Hence, the following corollary is immediate.

Corollary 1. Let $v \in TU^N$. If $\sigma \in \Pi(N)$ is such that

$$v([\sigma(i), \sigma] \cup S) + v([\sigma(k), \sigma]) \leq v([\sigma(i), \sigma]) + v([\sigma(k), \sigma] \cup S), \tag{10}$$

for all $0 \leq i < k \leq |N| - 1$ and $S \subseteq N\backslash[\sigma(k), \sigma]$ with $\sigma(k+1) \in S$, then $m^\sigma(v) \in C(v)$.

Let $v \in TU^N$ and $\sigma \in \Pi(N)$. Observe that the condition of Corollary 1 is weaker than that of Theorem 1 since in the condition of the corollary it is required that $S \subseteq N$ is such that $\sigma(k+1) \in S$. In fact, for each $0 \leq i < k \leq |N| - 1$, there are precisely $2^{|N|-k-1}$ choices of $S \subseteq N\backslash[\sigma(k), \sigma]$ such that $\sigma(k+1) \in S$. Therefore, Corollary 1 requires the checking of

$$\sum_{i=0}^{|N|-2} \sum_{k=i+1}^{|N|-1} 2^{|N|-k-1} = \sum_{i=0}^{|N|-2} [2^{|N|-i-1} - 1] = 2^{|N|} - 2 - (|N| - 1) = 2^{|N|} - |N| - 1$$

inequalities. Note that this is exactly the number of non-empty coalitions that are not heads[b] of σ. Furthermore observe that the number of inequalities in Corollary 1 is about half the number of permutational convexity inequalities. As a final remark we note that Corollary 1 can alternatively be proved literally the same as the proof of Theorem 1 in [2].

4 A refinement of permutational convexity

In this section we introduce a refinement of permutational convexity. We show that the conditions of this refinement are still sufficient for the corresponding marginal vector to be a core element.

Let $\sigma \in \Pi(N)$. Let $S \subseteq N$ and define $h(S) = \max\{j \in \{1, \ldots, |N|\} : \sigma(j) \notin S, S \not\subseteq [\sigma(j), \sigma]\}$ as the highest ordered player outside S that precedes at least one player in S. We remark that $h(S)$ only exists if S is not a head of σ.

We will call $v \in TU^N$ *weak permutationally convex with respect to* σ if for each $0 \leq i < k \leq |N| - 1$ and each $S \subseteq N\backslash[\sigma(k), \sigma]$ with $\sigma(k+1) \in S$ at least one of the following two inequalities is satisfied:

$$v(T) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup S) + v([\sigma(i), \sigma]) \tag{11}$$

$$v(T) + v([\sigma(h(S)), \sigma]) \leq v([\sigma(h(S)), \sigma] \cup T) + v(T \cap [\sigma(h(S)), \sigma]), \tag{12}$$

[b] A head of σ is a coalition $S \subseteq N$ such that $S = \{\sigma(1), \ldots, \sigma(k)\}$ for some $k \in \{1, \ldots, |N|\}$.

where $T = [\sigma(i), \sigma] \cup S$. Note that if S is connected, then the two inequalities coincide, since in this case $h(S) = k$.

Let $v \in TU^N$ and let $\sigma \in \Pi(N)$. Note that if the condition of Corollary 1 is satisfied for (N, v) and σ, then (N, v) is weak permutationally convex with respect to σ. This observation also implies that for weak permutational convexity there are precisely $2^{|N|} - |N| - 1$ pairs of inequalities.

In the following example we illustrate weak permutational convexity.

Example 4. Let $N = \{1, 2, 3, 4, 5\}$, $v \in TU^N$ and $\sigma \in \Pi(N)$ be given by $\sigma(i) = i$ for each $i \in \{1, 2, 3, 4, 5\}$. Let $i = 1$, $k = 2$ and $S = \{3, 5\}$. Then $h(S) = 4$. Hence, the corresponding condition for weak permutational convexity is $v(\{1, 3, 5\}) + v(\{1, 2\}) \leq v(\{1, 2, 3, 5\}) + v(\{1\})$ or $v(\{1, 3, 5\}) + v(\{1, 2, 3, 4\}) \leq v(N) + v(\{1, 3\})$. Note that if $i = 1$, $k = 3$ and $S = \{4, 5\}$, then $h(S) = 3 = k$. So both (11) and (12) boil down to $v(\{1, 4, 5\}) + v(\{1, 2, 3\}) \leq v(N) + v(\{1\})$.

The following theorem shows that weak permutational convexity is sufficient for the corresponding marginal vector to be a core element. We remark that this theorem implies Corollary 1 as well.

Theorem 3. Let $\sigma \in \Pi(N)$. If $v \in TU^N$ is weak permutationally convex with respect to σ, then $m^\sigma(v) \in C(v)$.

Proof: Since marginal vectors are efficient by definition, we only need to show $\sum_{i \in W} m_i^\sigma(v) \geq v(W)$ for each $W \subseteq N$. We will first show that $\sum_{i \in W} m_i^\sigma(v) \geq v(W)$ for each $W \subseteq N$ consisting of only one component. Then we show the inequality for each $W \subseteq N$ consisting of at least two components.

Let $W \subseteq N$ consist of only one component. If $\sigma(1) \in W$, then W is a head of σ, and trivially $\sum_{i \in W} m_i^\sigma(v) = v(W)$. So assume that $\sigma(1) \notin W$. Let $s \in N \backslash W$ be the highest ordered player in $N \backslash W$ preceding all players in W, and let $t \in W$ be the highest ordered player in W. Then

$$\sum_{i \in W} m_i^\sigma(v) = v([t, \sigma]) - v([s, \sigma]) \geq v(W).$$

The inequality is satisfied because both (11) and (12), with $i = 0$, $k = \sigma^{-1}(s)$ and $S = W$, coincide with $v(W) + v([s, \sigma]) \leq v([t, \sigma])$.

Now suppose that W consists of $a \geq 2$ components. Let $W_1, \ldots, W_a$ be these components. We assume that these components are ordered, i.e. if $1 \leq i < k \leq a$, then $W_i \subseteq [j, \sigma]$ for each $j \in W_k$. For each $1 \leq i \leq a$, let t_i be the highest ordered player in W_i, and let s_i be the highest ordered player in $N \backslash W$ preceding all players in W_i. Define $s_1 = 0$ in case $\sigma(1) \in W_1$. We will now show by induction on $q - p$ that

$$\sum_{l=p}^{q} \sum_{i \in W_l} m_i^\sigma(v) \geq v([s_p, \sigma] \cup \bigcup_{l=p}^{q} W_l) - v([s_p, \sigma]) \tag{13}$$

for all $1 \leq p < q \leq a$. First we show the induction basis. So let $1 \leq p < q \leq a$ be such that $q - p = 1$. Then,

$$\begin{aligned}
\sum_{l=p}^{q} \sum_{i \in W_l} m_i^\sigma(v) &= v([t_p, \sigma]) - v([s_p, \sigma]) + v([t_q, \sigma]) - v([s_q, \sigma]) \\
&\geq v([t_p, \sigma] \cup W_q) - v([s_p, \sigma]) \\
&= v([s_p, \sigma] \cup W_p \cup W_q) - v([s_p, \sigma]).
\end{aligned}$$

The inequality is satisfied because both (11) and (12), with $i = \sigma^{-1}(t_p)$, $k = \sigma^{-1}(s_q)$ and $S = W_q$, coincide with $v([t_p, \sigma] \cup W_q) + v([s_q, \sigma]) \leq v([t_q, \sigma]) + v([t_p, \sigma])$.

Now assume, as the induction hypothesis, that $1 \leq j \leq a - 1$ is such that (13) is satisfied for all $1 \leq p < q \leq a$ with $q - p \leq j$. If $j = a - 1$, then we are done, so assume that $j < a - 1$. For the induction step, let $1 \leq p^* < q^* \leq a$ with $q^* - p^* = j+1$. From (11) and (12) with $i = \sigma^{-1}(t_{p^*})$, $k = \sigma^{-1}(s_{p^*+1})$ and $S = \bigcup_{l=p^*+1}^{q^*} W_l$, we conclude that at least one of the following two inequalities is satisfied:

$$v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) + v([s_{p^*+1}, \sigma]) \leq v([s_{p^*+1}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) + v([t_{p^*}, \sigma]), \tag{14}$$

$$v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) + v([s_{q^*,\sigma}]) \leq v([t_{q^*}, \sigma]) + v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*-1} W_l). \tag{15}$$

First suppose that (14) is satisfied. Then,

$$\begin{aligned}
\sum_{l=p^*}^{q^*} \sum_{i \in W_l} m_i^\sigma(v) &= v([t_{p^*}, \sigma]) - v([s_{p^*}, \sigma]) + \sum_{l=p^*+1}^{q^*} \sum_{i \in W_l} m_i^\sigma(v) \\
&\geq v([t_{p^*}, \sigma]) - v([s_{p^*}, \sigma]) + v([s_{p^*+1}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) - v([s_{p^*+1}, \sigma]) \\
&\geq v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) - v([s_{p^*}, \sigma]) \\
&= v([s_{p^*}, \sigma] \cup \bigcup_{l=p^*}^{q^*} W_l) - v([s_{p^*}, \sigma]).
\end{aligned}$$

The first inequality is satisfied since, according to the induction hypothesis, (13) is satisfied for each $1 \leq p < q \leq a$ with $q - p \leq j$. In particular, (13) is satisfied for the pair p^*+1, q^*. The second inequality is due to (14).

Now suppose that (15) is satisfied. Then,

$$\begin{aligned}
\sum_{l=p^*}^{q^*} \sum_{i \in W_l} m_i^\sigma(v) &= \sum_{l=p^*}^{q^*-1} \sum_{i \in W_l} m_i^\sigma(v) + v([t_{q^*}, \sigma]) - v([s_{q^*}, \sigma]) \\
&\geq v([s_{p^*}, \sigma] \cup \bigcup_{l=p^*}^{q^*-1} W_l) - v([s_{p^*}, \sigma]) + v([t_{q^*}, \sigma]) - v([s_{q^*}, \sigma]) \\
&= v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) - v([s_{p^*}, \sigma]) + v([t_{q^*}, \sigma]) - v([s_{q^*}, \sigma]) \\
&\geq v([t_{p^*}, \sigma] \cup \bigcup_{l=p^*+1}^{q^*} W_l) - v([s_{p^*}, \sigma]) \\
&= v([s_{p^*}, \sigma] \cup \bigcup_{l=p^*}^{q^*} W_l) - v([s_{p^*}, \sigma]).
\end{aligned}$$

The first inequality is satisfied since, according to the induction hypothesis, (13) is satisfied for each $1 \leq p < q \leq a$ with $q - p \leq j$. In particular, (13) is satisfied for the pair $p^*, q^* - 1$. The second inequality is due to (15).

We conclude that (13) is satisfied for each $1 \leq p < q \leq a$. It follows from (13), with $p = 1$ and $q = a$, that

$$\sum_{i \in W} m_i^\sigma(v) \geq v([s_1, \sigma] \cup \bigcup_{l=1}^{a} W_l) - v([s_1, \sigma]) = v([s_1, \sigma] \cup W) - v([s_1, \sigma]). \tag{16}$$

If $s_1 = 0$, then we are done. So assume that $s_1 \neq 0$. In order to show that $\sum_{i \in W} m_i^\sigma(v) \geq v(W)$, we need to prove one more assertion with induction. To be more precise, we show for each $1 \leq r \leq a$ that

$$\sum_{l=1}^{r} \sum_{i \in W_l} m_i^\sigma(v) \geq v(\bigcup_{l=1}^{r} W_l). \tag{17}$$

This implies, using $r = a$, that

$$\sum_{i \in W} m_i^\sigma(v) = \sum_{l=1}^{r} \sum_{i \in W_l} m_i^\sigma(v) \geq v(\bigcup_{l=1}^{r} W_l) = v(W).$$

It remains to show that (17) is indeed satisfied for all $1 \leq r \leq a$. For the induction basis, let $r = 1$. Then,

$$\sum_{i \in W_1} m_i^\sigma(v) = v([t_1, \sigma]) - v([s_1, \sigma]) \geq v(W_1).$$

The inequality is satisfied because both (11) and (12), with $i = 0$, $k = \sigma^{-1}(s_1)$ and $S = W_1$, coincide with $v(W_1) + v([s_1, \sigma]) \leq v([t_1, \sigma])$.

Now assume, as the induction hypothesis, that $2 \leq j \leq a$ is such that (17) is satisfied for all $1 \leq r \leq j$. If $j = a$, then we are done, so assume that $j < a$. For the induction step, let $r^* = j + 1$. From (11) and (12), with $i = 0$, $k = \sigma^{-1}(s_1)$ and $S = \bigcup_{l=1}^{r^*} W_l$, we conclude that at least one of the following two inequalities is satisfied:

$$\begin{aligned} v(\bigcup_{l=1}^{r^*} W_l) + v([s_1, \sigma]) &\leq v([s_1, \sigma] \cup \bigcup_{l=1}^{r^*} W_l) && (18) \\ v(\bigcup_{l=1}^{r^*} W_l) + v([s_{r^*}, \sigma]) &\leq v([t_{r^*}, \sigma]) + v(\bigcup_{l=1}^{r^*-1} W_l). && (19) \end{aligned}$$

First suppose that (18) is satisfied. Then,

$$\sum_{l=1}^{r^*} \sum_{i \in W_l} m_i^\sigma(v) \geq v([s_1, \sigma] \cup \bigcup_{l=1}^{r^*} W_l) - v([s_1, \sigma]) \geq v(\bigcup_{l=1}^{r^*} W_l).$$

The first inequality follows from it follows from (13) with $p = 1$ and $q = r^*$ that and the second from (18).

Now suppose that (19) is satisfied. Then

$$\begin{aligned} \sum_{l=1}^{r^*} \sum_{i \in W_l} m_i^\sigma(v) &= \sum_{l=1}^{r^*-1} \sum_{i \in W_l} m_i^\sigma(v) + v([t_{r^*}, \sigma]) - v([s_{r^*}, \sigma]) \\ &\geq v(\bigcup_{l=1}^{r^*-1} W_l) + v([t_{r^*}, \sigma]) - v([s_{r^*}, \sigma]) \\ &\geq v(\bigcup_{l=1}^{r^*} W_l). \end{aligned}$$

The first inequality is due to our induction hypothesis that (17) is satisfied for all $r \leq j$. In particular, (17) is satisfied for $r = r^* - 1$. The second inequality holds because of (19). □

The following example is meant to illustrate that weak permutational convexity is not a necessary condition for the corresponding marginal vector to be a core element.

Example 5. Consider $v \in TU^N$ with $N = \{1,2,3,4\}$, $v(S) = 0$ if $S \in 2^N \backslash \{\{1,2\},\{2,4\},$ $\{1,2,3\}, N\}$ and $v(\{1,2\}) = v(\{2,4\}) = v(\{1,2,3\}) = v(N) = 1$. Let $\sigma \in \Pi(N)$ be given by $\sigma(i) = i$ for each $i \in \{1,2,3,4\}$. Observe that (N,v) is not weak permutationally convex with respect to σ, since the corresponding condition is not satisfied for $i = 0$, $k = 1$ and $S = \{2,4\}$. However, $m^\sigma(v) = (0,1,0,0) \in C(v)$. $\diamond$

5 A restricted set of inequalities

In this section we obtain a reduced set of inequalities that is equivalent to permutational convexity. Furthermore we give conditions such that if a game is permutationally convex with respect to one order, then it is permutationally convex with respect to another order as well. As a corollary we obtain that if a game is permutationally convex with respect to an order, then it is also permutationally convex with respect to the last neighbour of this order.

The following proposition shows that permutational convexity is equivalent to a restricted set of inequalities.

Proposition 1. Let $v \in TU^N$. Then (N,v) is permutationally convex with respect to $\sigma \in \Pi(N)$ if and only if (1) is satisfied for all $0 \leq i < k \leq |N|-1$ with $i+1 = k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$.

Proof: The "only if" part follows directly from the definition of permutational convexity. Therefore we only show the "if" part. Assume that (1) is satisfied for all $0 \leq i < k \leq |N|-1$ with $i+1 = k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. It remains to show that (1) is also satisfied for all $0 \leq i < k \leq |N|-1$ with $i+1 \neq k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. We use backwards induction on i.

For the induction basis let $i = |N|-3$, $k = |N|-1$ and $S \subseteq N \backslash [\sigma(|N|-1), \sigma]$ with $S \neq \emptyset$. Hence, $S = \{\sigma(|N|)\}$. We know from our initial assumption that (1) is satisfied for $\bar{i} = |N|-2$, $\bar{k} = |N|-1$ and $S = \{\sigma(|N|)\}$. Hence,

$$v([\sigma(|N|-2), \sigma] \cup \{\sigma(|N|)\}) + v([\sigma(|N|-1), \sigma]) \leq v(N) + v([\sigma(|N|-2), \sigma]). \tag{20}$$

Furthermore, our initial assumption implies that (1) is satisfied for $\hat{i} = |N|-3$, $\hat{k} = |N|-2$ and $S = \{\sigma(|N|)\}$. Therefore,

$$v([\sigma(|N|-3), \sigma] \cup \{\sigma(|N|)\}) + v([\sigma(|N|-2), \sigma]) \leq v([\sigma(|N|-2), \sigma] \cup \{\sigma(|N|)\}) + v([\sigma(|N|-3), \sigma]). \tag{21}$$

Adding (20) and (21) yields

$$v([\sigma(|N|-3), \sigma] \cup \{\sigma(|N|)\}) + v([\sigma(|N|-1), \sigma]) \leq v(N) + v([\sigma(|N|-3), \sigma]),$$

which is precisely (1) with $i = |N|-3$, $k = |N|-1$ and $S = \{\sigma(|N|)\}$.

For the induction hypothesis we assume that for some $1 \leq i^* \leq |N|-3$ we have shown that (1) is satisfied for all $0 \leq i < k \leq |N|-1$ with $i^* \leq i$, $i+1 < k$, and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$.

For the induction step, let $0 \leq i < k \leq |N|-1$ be such that $i = i^*-1$, $i+1 < k$ and $S \subseteq N \backslash [\sigma(k), \sigma]$ with $S \neq \emptyset$. From our induction hypothesis it follows that (1) is satisfied for $\bar{i} = i^*$, $\bar{k} = k$ and S. Hence,

$$v([\sigma(i^*), \sigma] \cup S) + v([\sigma(k), \sigma]) \leq v([\sigma(k), \sigma] \cup S) + v([\sigma(i^*), \sigma]). \tag{22}$$

We also know from our initial assumption that (1) is satisfied for $\bar{i} = i$, $\bar{k} = i^*$ and S, since $\bar{i}+1 = \bar{k}$. This yields

$$v([\sigma(i), \sigma] \cup S) + v([\sigma(i^*), \sigma]) \leq v([\sigma(i^*), \sigma] \cup S) + v([\sigma(i), \sigma]). \tag{23}$$

Now adding (22) and (23) gives

$$v([\sigma(i),\sigma]\cup S)+v([\sigma(k),\sigma])\leq v([\sigma(k),\sigma]\cup S)+v([\sigma(i),\sigma]),$$

which is precisely the inequality we needed to show. $\square$

Let $v\in TU^N$ and $\sigma\in\Pi(N)$. According to Proposition 1, permutational convexity of σ requires the checking of precisely $2^{|N|-k}-1$ inequalities for each pair $0\leq i<k\leq |N|-1$ with $i+1=k$. This implies that in total

$$\sum_{k=1}^{|N|-1}[2^{|N|-k}-1]=2^{|N|}-2-(|N|-1)=2^{|N|}-|N|-1$$

inequalities need to be checked. In particular, Proposition 1 reduces the number of permutational convexity inequalities by a factor two.

The following proposition presents inequalities such that if a game is permutationally convex with respect to $\sigma\in\Pi(N)$, then it is permutationally convex with respect to a neighbour of σ as well.

Proposition 2. Let $v\in TU^N$. Let $\sigma\in\Pi(N)$ be a permutationally convex order for (N,v) and let $1\leq l\leq |N|-1$. If

$$v([\sigma_l(i),\sigma_l]\cup S)+v([\sigma_l(k),\sigma_l])\leq v([\sigma_l(k),\sigma_l]\cup S)+v([\sigma_l(i),\sigma_l]), \tag{24}$$

for $i=l-1$, $k=l$ and all $S\subseteq N\backslash[\sigma_l(l+1),\sigma_l]$ with $S\neq\emptyset$, and for $i=l$, $k=l+1$ and all $S\subseteq N\backslash[\sigma_l(l+1),\sigma_l]$ with $S\neq\emptyset$, then σ_l is permutationally convex for (N,v).

Proof: According to Proposition 1 showing that σ_l is permutationally convex boils down to showing (24) for all $0\leq i<k\leq |N|-1$ with $i+1=k$ and $S\subseteq N\backslash[\sigma_l(k),\sigma_l]$ with $S\neq\emptyset$. We distinguish between three cases.

`Case 1:` $i\leq l-2$ or $i\geq l+1$.

In this case $[\sigma_l(i),\sigma_l]=[\sigma(i),\sigma]$ and $[\sigma_l(k),\sigma_l]=[\sigma(k),\sigma]$. Let $S\subseteq N\backslash[\sigma_l(k),\sigma_l]=N\backslash[\sigma(k),\sigma]$ with $S\neq\emptyset$. Since σ is permutationally convex for (N,v) we conclude that (1) is satisfied for i, $k=i+1$ and S. Observe that (1) coincides with (24).

`Case 2:` $i=l-1$.

In this case $k=i+1=l$. We need to show that (24) holds for all $S\subseteq N\backslash[\sigma_l(k),\sigma_l]$ with $S\neq\emptyset$. First note that if $\sigma_l(k+1)\notin S$, then $S\subseteq N\backslash[\sigma_l(k+1),\sigma_l]=N\backslash[\sigma_l(l+1),\sigma_l]$. In this case (24) is satisfied by assumption. So now suppose that $\sigma_l(k+1)\in S$. Let $T=S\backslash\{\sigma_l(k+1)\}$. Since σ is permutationally convex for (N,v) it follows that (1) holds with $\bar{i}=i$, $\bar{k}=k$ and $\bar{S}=\{\sigma(k+1)\}$, i.e.,

$$v([\sigma(i),\sigma]\cup\{\sigma(k+1)\})+v([\sigma(k),\sigma])\leq v([\sigma(k),\sigma]\cup\{\sigma(k+1)\})+v([\sigma(i),\sigma]),$$

which is equivalent to

$$v([\sigma(i),\sigma]\cup\{\sigma(k+1)\})+v([\sigma(k),\sigma])\leq v([\sigma(k+1),\sigma])+v([\sigma(i),\sigma]). \tag{25}$$

Furthermore, (1) is satisfied for $\hat{i}=i+1=k$, $\hat{k}=k+1$ and $\hat{S}=T$ since σ is permutationally convex by assumption. This implies

$$v([\sigma(k),\sigma]\cup T)+v([\sigma(k+1),\sigma])\leq v([\sigma(k+1),\sigma]\cup T)+v([\sigma(k),\sigma]). \tag{26}$$

Now adding (25) and (26) yields

$$v([\sigma(k),\sigma]\cup T)+v([\sigma(i),\sigma]\cup\{\sigma(k+1)\})\leq v([\sigma(k+1),\sigma]\cup T)+v([\sigma(i),\sigma]). \tag{27}$$

Observe that $[\sigma(k),\sigma]\cup T=[\sigma_l(i),\sigma_l]\cup S$, $[\sigma(i),\sigma]\cup\{\sigma(k+1)\}=[\sigma_l(k),\sigma_l]$, $[\sigma(k+1),\sigma]\cup T=[\sigma_l(k),\sigma_l]\cup S$ and $[\sigma(i),\sigma]=[\sigma_l(i),\sigma_l]$. This shows that (27) coincides with (24).

`Case 3:` $i=l$.

In this case $k=i+1=l+1$. Let $S\subseteq N\backslash[\sigma_l(l+1),\sigma_l]$ with $S\neq\emptyset$. Now (24) is satisfied by assumption. □

Proposition 2 easily constitutes the following corollary. This corollary shows that if one order is permutationally convex, then its last neighbour is permutationally convex as well. Hence, for each game an even number of orders is permutationally convex.

Corollary 2. Let $v\in TU^N$. If $\sigma\in\Pi(N)$ is permutationally convex for (N,v), then $\sigma_{|N|-1}$ is permutationally convex for (N,v) as well.

Proof: According to Proposition 2 it is sufficient to show that

$$v([\sigma_{|N|-1}(i),\sigma_{|N|-1}]\cup S)+v([\sigma_{|N|-1}(k),\sigma_{|N|-1}])\leq v([\sigma_{|N|-1}(k),\sigma_{|N|-1}]\cup S)+v([\sigma_{|N|-1}(i),\sigma_{|N|-1}])$$

is satisfied for $i=|N|-2$, $k=|N|-1$ and all $S\subseteq N\backslash[\sigma_{|N|-1}(|N|),\sigma_{|N|-1}]$ with $S\neq\emptyset$, and for $i=|N|-1$, $k=|N|$ and all $S\subseteq N\backslash[\sigma_{|N|-1}(|N|),\sigma_{|N|-1}]$ with $S\neq\emptyset$. However, if $S\subseteq N\backslash[\sigma_{|N|-1}(|N|),\sigma_{|N|-1}]$, then $S=\emptyset$. □

Appendix

In this section we state and prove the technical lemmas that are needed for the proof of Theorem 2.

Lemma 1. It holds that $B(E(Q))=E(Q)$, $A(E(Q))=\bigcup_{q\in E(Q)}D(q)$ and $E(Q)\cup\bigcup_{i\in A(E(Q))}P(i)=\bigcup_{q\in E(Q)}P(q)$.

Proof: We first show $B(E(Q))=E(Q)$. By definition, $B(E(Q))\subseteq E(Q)$. Now let $i\in E(Q)$. Then $i\notin P(q)$ for each $q\in Q\backslash\{i\}$. Hence, $P(i)\cap Q=\{i\}$ for each $i\in E(Q)$, and therefore also $P(i)\cap E(Q)=\{i\}$ for each $i\in E(Q)$. So $i\in B(E(Q))$ and we conclude that $E(Q)\subseteq B(E(Q))$.

Observe that $A(E(Q))=\bigcup_{q\in B(E(Q))}D(q)=\bigcup_{q\in E(Q)}D(q)$. The first equality is satisfied by definition of $A(E(Q))$, and the second because $B(E(Q))=E(Q)$.

It remains to show that $E(Q)\cup\bigcup_{i\in A(E(Q))}P(i)=\bigcup_{q\in E(Q)}P(q)$. Observe that

$$E(Q)\cup\bigcup_{i\in A(E(Q))}P(i)=E(Q)\cup\bigcup_{i\in\bigcup\limits_{q\in E(Q)}D(q)}P(i)=\bigcup_{q\in E(Q)}P(q).$$

The first equality is satisfied because $A(E(Q))=\bigcup_{q\in E(Q)}D(q)$. □

Lemma 2. It holds that $A(T_{t^*-1})\cup\bigcup_{q\in DP(l)}D(q)=A(T_{t^*})\cup D(l)$.

Proof: Observe that

$$\begin{aligned}
A(T_{t^*-1}) \cup \bigcup_{q\in DP(l)} D(q) &= \bigcup_{q\in B(T_{t^*-1})} D(q) \cup \bigcup_{q\in DP(l)} D(q) \\
&= \bigcup_{q\in B(T_{t^*-1})\setminus\{l\}} D(q) \cup D(l) \cup \bigcup_{q\in DP(l)} D(q) \\
&= \bigcup_{q\in B(T_{t^*})} D(q) \cup D(l) \\
&= A(T_{t^*}) \cup D(l).
\end{aligned}$$

The first equality holds by definition of $A(T_{t^*-1})$. For the second equality we have used that $l \in B(T_{t^*-1})$. The third equality follows from $(B(T_{t^*-1})\setminus\{l\}) \cup DP(l) = B(T_{t^*})$. The last equality follows again by definition of $A(T_{t^*})$. □

Lemma 3. It holds that $\sum_{i\in A(T_{t^*-1})} v(P(i)) + \sum_{q\in DP(l)} \sum_{i\in D(q)} v(P(i)) = \sum_{i\in A(T_{t^*})} v(P(i)) + \sum_{i\in D(l)} v(P(i))$.

Proof: Follows directly from Lemma 2. □

Lemma 4. It holds that

$$\bigcup_{i\in A(T_{t^*-1})} P(i) = \bigcup_{i\in A(T_{t^*-1})\setminus D(l)} P(i) \cup \bigcup_{i\in D_1(l)} P(i) \cup \bigcup_{i\in D_2(l)} P(i) \cup \bigcup_{i\in D_3(l)} P(i)$$

and that

$$T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})} P(i) = \bar{T} \cup \bigcup_{i\in D_2(l)} P(i) \cup \bigcup_{i\in D_3(l)} P(i)$$

with $\bar{T} = T_{t^*-1} \cup \bigcup_{i\in A(T_{t^*-1})\setminus D(l)} P(i) \cup \bigcup_{i\in D_1(l)} P(i)$.

Proof: We only show the first assertion, since the second assertion is a consequence of the first one. Note that

$$\begin{aligned}
\bigcup_{i\in A(T_{t^*-1})} P(i) &= \bigcup_{i\in A(T_{t^*-1})\setminus D(l)} P(i) \cup \bigcup_{i\in D(l)} P(i) \\
&= \bigcup_{i\in A(T_{t^*-1})\setminus D(l)} P(i) \cup \bigcup_{i\in D_1(l)} P(i) \cup \bigcup_{i\in D_2(l)} P(i) \cup \bigcup_{i\in D_3(l)} P(i).
\end{aligned}$$

The first equality is satisfied because $l \in B(T_{t^*-1})$. Hence, $D(l) \subseteq A(T_{t^*-1})$ and $A(T_{t^*-1}) = (A(T_{t^*-1})\setminus D(l)) \cup D(l)$. The second equality follows by definition of $D_1(l)$, $D_2(l)$ and $D_3(l)$. □

Lemma 5. The sets $M_l(q)$, $q \in D_3(l)\}$, form a partition of $DP(l)\setminus D_1(l)$.

Proof: Because $P(i) \cap P(j) = \emptyset$ for each $i, j \in D_3(l)$, $i \neq j$, it follows that $M_l(i) \cap M_l(j) = \emptyset$ for each $i, j \in D_3(l)$, $i \neq j$.

Now let $q \in D_3(l)$ and $i \in M_l(q)$. Then $i \in DP(l)$. Since $q \notin D_1(l)$ and $i \in P(q)$, it follows that $i \notin D_1(l)$. So $i \in DP(l)\setminus D_1(l)$. We conclude that $M_l(q) \subseteq DP(l)\setminus D_1(l)$ for each $q \in D_3(l)$.

It remains to show that $\bigcup_{q\in D_3(l)} M_l(q) = DP(l)\setminus D_1(l)$. Since $M_l(q) \subseteq DP(l)\setminus D_1(l)$ for each $q \in D_3(l)$, it is sufficient to show that $DP(l)\setminus D_1(l) \subseteq \bigcup_{q\in D_3(l)} M_l(q)$. Let $i \in DP(l)\setminus D_1(l)$. Then there is a $q \in D_3(l)$ with $i \in P(q)$. Hence, $i \in DP(l) \cap P(q) = M_l(q)$. □

Lemma 6. It holds that $\sum_{q\in DP(l)\setminus D_1(l)} v(P(q)) - \sum_{q\in D_3(l)} v(R(q)) \geq 0$.

Proof: Because of Lemma 5, $\{M_l(q) : q \in D_3(l)\}$ forms a partition of $DP(l) \backslash D_1(l)$. Therefore

$$\sum_{q \in DP(l) \backslash D_1(l)} v(P(q)) = \sum_{q \in D_3(l)} \sum_{i \in M_l(q)} v(P(i)).$$

Now let $q \in D_3(l)$ and $i, k \in M_l(j)$, $i \neq k$. Since $i, k \in DP(l) \cap P(q)$, it follows that $i, k \in DP(l)$. It follows that $P(i) \cap P(k) = \emptyset$. We conclude that for each $j \in D_3(l)$ and each $i, k \in M_l(j)$, $i \neq k$, that $P(i) \cap P(k) = \emptyset$. Hence, by applying (2) with $M = M_l(q)$ for each $q \in D_3(l)$ we find that

$$\sum_{i \in M_l(q)} v(P(i)) \geq v(\bigcup_{i \in M_l(q)} P(i)) = v(R(q)).$$

Summing over all $q \in D_3(l)$ gives

$$\sum_{q \in DP(l) \backslash D_1(l)} v(P(q)) = \sum_{q \in D_3(l)} \sum_{i \in M_l(q)} v(P(i)) \geq \sum_{q \in D_3(l)} v(R(q)). \qquad \square$$

Lemma 7. It holds that $\bar{T} \cup \bigcup_{q \in D_3(l)} R(q) = T_{t^*} \cup \bigcup_{i \in A(T_{t^*})} P(i)$.

Proof: Because $\bar{T} = T_{t^*-1} \cup \bigcup_{i \in A(T_{t*-1}) \backslash D(l)} P(i) \cup \bigcup_{i \in D_1(l)} P(i)$ and because $T_{t^*} = T_{t^*-1} \cup DP(l)$ it is sufficient to show that

$$\bigcup_{i \in A(T_{t^*-1}) \backslash D(l)} P(i) \cup \bigcup_{i \in D_1(l)} P(i) \cup \bigcup_{i \in D_3(l)} R(i) = DP(l) \cup \bigcup_{i \in A(T_{t^*})} P(i). \tag{28}$$

First note that it follows from Lemma 2, and the observation that $A(T_{t^*}) \cap D(l) = \emptyset$, that $A(T_{t^*}) = (A(T_{t^*-1}) \backslash D_1(l)) \cup \bigcup_{q \in DP(l)} D(q)$. This implies that $\bigcup_{i \in A(T_{t^*})} P(i) = \bigcup_{i \in A(T_{t^*-1}) \backslash D(l)} P(i) \cup \bigcup_{i \in D(q) : q \in DP(l)} P(i)$. Combining this equality with (28), we conclude that it is sufficient to show that

$$\bigcup_{i \in D_1(l)} P(i) \cup \bigcup_{i \in D_3(l)} R(i) = DP(l) \cup \bigcup_{i \in D(q) : q \in DP(l)} P(i).$$

We will show this equality by first rewriting $\bigcup_{i \in D_3(l)} R(i)$. Observe that

$$\bigcup_{i \in D_3(l)} R(i) = \bigcup_{i \in D_3(l)} \left(\bigcup_{q \in M_l(i)} P(q) \right) = \bigcup_{i \in M_l(q) : q \in D_3(l)} P(i) = \bigcup_{i \in DP(l) \backslash D_1(l)} P(i).$$

The first equality is satisfied by definition of $R(i)$. The last equality is satisfied because $\{M_l(q) : q \in D_3(l)\}$ forms a partition of $DP(l) \backslash D_1(l)$. It is now straightforward to verify that

$$\begin{aligned} \bigcup_{i \in D_1(l)} P(i) \cup \bigcup_{i \in D_3(l)} R(i) &= \bigcup_{i \in D_1(l)} P(i) \cup \bigcup_{i \in DP(l) \backslash D_1(l)} P(i) \\ &= \bigcup_{i \in DP(l)} P(i) \\ &= DP(l) \cup \bigcup_{i \in D(q) : q \in DP(l)} P(i). \qquad \square \end{aligned}$$

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